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Fractional diffusion equation on fractals: three-dimensional case and scattering function

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Abstract. A fractional equation for diffusion in isotropic and homogeneous fractal structures is discussed. It generalizes the fractional diffusion equation valid for d-dimensional Euclidean systems. The asymptotic behaviour of the probability density function is obtained exactly. Analytical expressions are derived for the scattering and relaxation functions, which can be studied by x-ray and neutron scattering experiments on fractals.

1. Introduction

In a previous work [1], an approximate fractional differential equation describing the asymptotic behaviour of diffusion on fractal structures has been proposed. The solution of the fractional diffusion equation (FDE) correctly describes the stretched Gaussian shape of the probability density function $P(r,t)$ (when $r \to \infty$), valid on a large class of fractal structures [2–5]. The approximate form of the FDE discussed in [1], however, reduces exactly to its standard counterpart for one-dimensional systems only.

In this paper we study a more general form of the FDE which reduces exactly to its standard counterpart in d-dimensional Euclidean systems. To obtain the new FDE, we discuss first the general form of the fractional diffusion equation in the d-dimensional Euclidean case.

In the case of an isotropic and homogeneous medium, the density function $P(r,t)$ giving the probability that a Brownian particle is at distance r at time t, obeys the diffusion equation

$$\frac{\partial P(r,t)}{\partial t} = \frac{D_0}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial P(r,t)}{\partial r}\right) \tag{1.1}$$

where D_0 is the diffusion coefficient and d the Euclidean dimension of the system. The solution of (1.1) is a Gaussian, $P(r,t) \sim t^{-d/2}\exp[-\text{constant}(r/R)^2]$, and the mean-square displacement of the Brownian particle $\langle r^2(t)\rangle \equiv R^2 \cong D_0\, t$.

Denoting by $P(r,s)$ the Laplace transform of $P(r,t)$,

$$P(r,s) = \int_0^\infty \mathrm{d}t\, \exp(-st)P(r,t)$$

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and assuming $D_0 = 1$ for simplicity, (1.1) can be written in the Laplace domain as

$$sP(r,s) = \frac{\partial^2 P(r,s)}{\partial r^2} + \frac{d-1}{r}\frac{\partial P(r,s)}{\partial r}. \tag{1.2}$$

According to Oldham and Spanier [6] (see also [1]), the fractional equation corresponding to (1.1) relates the time derivative of order $\frac{1}{2}$ of $P(r,t)$, defined as

$$\partial_t^{1/2} P(r,t) = \frac{1}{\Gamma(1-\frac{1}{2})}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{P(r,\tau)}{(t-\tau)^{1/2}}\,\mathrm{d}\tau \tag{1.3}$$

to its spatial derivative as follows

$$\partial_t^{1/2} P(r,t) = -\sqrt{D}_0\left(\frac{\partial P(r,t)}{\partial r} + \frac{\kappa}{r}P(r,t)\right). \tag{1.4}$$

To determine the coefficient κ, it is convenient to define the 'fractional' diffusion operator $\hat{H}$ as

$$\hat{H} = -\left(\frac{\partial}{\partial r} + \frac{\kappa}{r}\right)$$

and (1.4) can be written, in the Laplace domain (see [6]), as $s^{1/2}P(r,s) = \hat{H}P(r,s)$ (when $D_0 = 1$). Applying $\hat{H}$ to the last equation from the left, we obtain

$$sP(r,s) = \frac{\partial^2 P(r,s)}{\partial r^2} + \frac{2\kappa}{r}\frac{\partial P(r,s)}{\partial r} + \frac{\kappa(\kappa-1)}{r^2}P(r,s). \tag{1.5}$$

We see that (1.5) coincides with (1.2) up to terms of order $1/r$ provided that

$$\kappa = (d-1)/2. \tag{1.6}$$

When $d = 1$ and $d = 3$, i.e. $\kappa = 0$ and $\kappa = 1$ respectively, (1.5) is identical to (1.2), and the fractional diffusion equation (1.4) yields the exact result. In two dimensions, however, (1.4) is valid only asymptotically, $r/R \to \infty$ [6].

In the following, we generalize (1.4) for describing diffusion on fractals according to the procedure discussed in [1].

2. Fractional diffusion equation on fractals

Diffusion in complex media such as fractals displays several anomalous features: the mean-square displacement of a Brownian particle is slowed down on all time scales obeying

$$R^2 \sim t^{2/d_w} \tag{2.1}$$

where $d_w > 2$ is the anomalous diffusion exponent, and the probability density function $P(r,t)$ displays a non-Gaussian shape [2–5].

To describe $P(r,t)$ on isotropic and homogeneous fractals, we study a generalized fractional diffusion equation obtained from (1.3) by identifying the exponent $\frac{1}{2}$ in the time derivative of $P(r,t)$ with its anomalous counterpart $1/d_w$, i.e.

$$\partial_t^{1/d_w} P(r,t) = \frac{1}{\Gamma(1-1/d_w)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{P(r,\tau)}{(t-\tau)^{1/d_w}} \, \mathrm{d}\tau.$$

Thus, (1.4) becomes

$$\partial_t^{1/d_w} P(r,t) = -A \left(\frac{\partial P(r,t)}{\partial r} + \frac{\kappa}{r} P(r,t) \right) \tag{2.2}$$

where $A > 0$ is a constant. To determine κ on fractals, it is useful to recall the geometrical meaning of the r-dependent factors in (1.1).

On regular systems, the factor r^{d-1} in (1.1) represents the area of the hypersurface in a d-dimensional space (or number of sites on a lattice) available for diffusion at a distance r from the origin. From this follows the result (1.6). On fractals, the available 'area' grows as r^{d_f-1}, where d_f is the fractal dimension. Thus, purely geometric considerations would lead us to replace d by d_f in (1.6) for fractals, and $\kappa = (d_f - 1)/2$ would be obtained. We will find, however, that this simple choice may not be the appropriate one.

The solution of (2.2) in the Laplace domain reads

$$P(r,s) = Q(s) \frac{1}{(rs^{1/d_w})^\kappa} \exp(-rs^{1/d_w}/A) \tag{2.3}$$

where $Q(s)$ can be determined from the normalization condition (see [1])

$$\int_0^\infty \mathrm{d}V(r)\, P(r,s) = 1/s \qquad \mathrm{d}V(r) = \Lambda \mathrm{d}r\, r^{d_f-1} \qquad \Lambda > 0$$

leading to

$$Q(s) = \frac{A'}{s^{1-d_s/2}} \qquad A' > 0$$

where $d_s = 2d_f/d_w$ is the spectral or fracton dimension [7]. It is easy to see that (2.3) is consistent with the anomalous behaviour (2.1), since $\int_0^\infty \mathrm{d}V(r)\, r^2 P(r,s) \sim s^{-(1+2/d_w)}$ corresponds to the Laplace transform of $\langle r^2(t) \rangle$.

To learn about the behaviour of the probability density in the temporal domain, it is instructive to consider first the behaviour of $P(r,t)$ when $r/R \to \infty$. As discussed in appendix A, the probability density obeys the *stretched* Gaussian form asymptotically

$$P(r,t) \cong P_0(t) \left(\frac{r}{R}\right)^\alpha \exp[-b(r/R)^u] \qquad r/R \gg 1 \tag{2.4}$$

where

$$u = \frac{d_w}{d_w - 1} < 2 \tag{2.5}$$

$$\alpha = \frac{u}{2}(d_s - 1 - 2\kappa) \tag{2.6}$$

and

$$P_0(t) = \frac{a}{t^{d_s/2}} \qquad (2.7)$$

ensures the normalization condition of $P(r,t)$.

Our result (2.4) is in agreement with the stretched Gaussian form of $P(r,t)$ on fractals [2–5]. It is interesting to note that the value $\alpha = 0$ is not inconsistent with the presently available exact enumeration results [3, 4]. Such a value is obtained when

$$\kappa = \frac{d_s - 1}{2} \qquad (2.8)$$

(see equation 2.6). This would mean physically that information about the dynamic process (contained in the exponent d_w) is required for determining the exponent κ. In contrast, in the Euclidean case κ has a purely geometrical origin.

Having discussed the asymptotic behaviour of $P(r,t)$ as obtained from (2.3), we consider next the behaviour of $P(r,t)$ near the origin, $r = 0$. As shown in appendix B, (2.3) can be inverted and a closed integral form of $P(r,t)$ can be obtained analytically. We find that near the origin $P(r,t)$ displays the power-law behaviour

$$P(r,t) \sim P_0(t) \left(\frac{r}{R}\right)^{-\kappa} \qquad r/R \ll 1 \qquad (2.9)$$

governed by the exponent κ. When $\kappa > 0$, as in our case (equation (2.8)), $P(r,t)$ displays a weak (integrable) singularity near the origin. This result is in strong contrast with the standard behaviour of $P(r,t)$ on Euclidean systems. Although (2.9) strictly holds in the continuum (on the lattice $P(0,t)$ is finite because of the cutoff imposed by the lattice spacing a), it might still be detected in numerical simulations on the lattice for sufficiently long times t, such that $R(t) \gg 1$ and a wide interval of distances $a < r < R$ can be studied. Accurate numerical work along this direction is therefore highly desirable.

In the following, we calculate the scattering function $S(k,\omega)$ corresponding to the present diffusion problem. This quantity is of interest because it can be studied by x-ray and neutron scattering experiments on fractals. As a by-product, we obtain the asymptotic behaviour of the relaxation function $P(k,t)$.

3. Scattering and relaxation function for diffusion on fractals

The scattering function, or dynamic structure factor [8], $S(k,\omega)$ can be obtained from the relation

$$S(k,\omega) = \mathrm{Re}\, S(k,s) \qquad s = \mathrm{i}\omega \qquad (3.1)$$

where

$$S(k,s) = \int \mathrm{d}r\, \exp(\mathrm{i}k\cdot r) \int_0^\infty \mathrm{d}t\, \exp(-st) P(r,t) \qquad (3.2)$$

can be evaluated exactly using the Laplace transform result (2.3). Considering a three-dimensional embedding space and after integrating over the angles, (3.2) becomes

$$S(k,s) = \frac{2\pi A'}{s^{1-d_s/2}} \int_0^\infty \mathrm{d}r\, r^{d_f-1} \frac{\sin(kr)}{kr} \frac{1}{(rs^{1/d_w})^\kappa} \exp(-rs^{1/d_w}/A)$$

whose solution is

$$S(k,s) = \frac{2\pi A'}{s(ks^{-1/d_w})^{d_f-\kappa}} \frac{\Gamma(\mu)}{(\beta^2+1)^{\mu/2}} \sin\left[\mu \tan^{-1}(1/\beta)\right] \qquad (3.4)$$

where $\beta = s^{1/d_w}/(kA)$ and $\mu = d_f - \kappa - 1$. We note that according to (3.4), $S(k,\omega)$ obeys the scaling form

$$S(k,\omega) = \frac{1}{k^{d_w}} g(\omega/\omega_k)$$

where $g(x)$ is the scaling function and

$$\omega_k \sim k^{d_w} \qquad (3.5)$$

is the characteristic frequency. To obtain the asymptotic behaviour of $S(k,\omega)$, let us consider first the case $\omega \gg \omega_k$, which corresponds to $\beta \gg 1$ in (3.4). In this limit, we obtain after some algebra

$$S(k,\omega) \sim B\, \frac{A^2k^2}{\omega^\delta} \qquad \omega \gg \omega_k \qquad (3.6a)$$

where

$$\delta = 1 + \frac{2}{d_w} \qquad (3.6b)$$

and $B = 2\pi A'\Gamma(\mu)\sin(\pi/d_w)(d_f-\kappa)[(d_f-\kappa)^2-1]/6$. It should be emphasized that the value of δ thus obtained (equation (3.6*b*)) is independent of the choice of κ. The result (3.6) is quite robust because it corresponds to the asymptotic behaviour of $P(r,t)$, $r/R \to \infty$, which is described accurately by the present approach, equation (2.2). Although the result (3.6) is not new (see e.g. [9]†), we do not know of any rigorous derivation of it.

Let us consider now the case $\omega \ll \omega_k$, i.e. the case $\beta \to 0$ in (3.4). This limit corresponds to the behaviour of $P(r,t)$ when $r/R \to 0$. Here, we find a weak power-law divergency when $\omega \to 0$ of the form

$$S(k,\omega) \sim \frac{1}{\omega^\gamma} \frac{1}{k^{d_w(1-\gamma)}} \qquad \omega \ll \omega_k \qquad (3.7a)$$

where

$$\gamma = 1 - \frac{d_f - \kappa}{d_w}. \qquad (3.7b)$$

† Although the authors of [9] do not consider the problem of diffusion on fractals, the asymptotic behaviour (3.6) is derived from quite general assumptions.

It should be emphasized that even in the case that $\kappa = 0$, implying a smooth behaviour of $P(r,t)$ near $r = 0$, a singularity in the structure factor (equation (3.7)) is still present near $\omega = 0$ since $\gamma = 1 - d_f/d_w > 0$ in this case. Note that (3.6) and (3.7) correspond to the Lorentzian result (see e.g. [8])

$$S(k,\omega) = \frac{2D_0k^2}{\omega^2 + (D_0k^2)^2}$$

with $\delta = 2$ and $\gamma = 0$, when $d_s = d_f = d = 3$ and $d_w = 2$, as required.

The structure factor $S(k,\omega)$ obtained from (3.1) and (3.4) is plotted in figure 1 as a function of ω, when $k = 1$. The asymptotic behaviours of (3.6) and (3.7) are displayed by the straight lines. The crossover frequency ω_k may be also obtained graphically as indicated in the figure. Their expected k-dependence (3.5) has been verified numerically. Figure 2 shows $S(k,\omega)$ against frequency ω for several values of d_w, including the Lorentzian result.

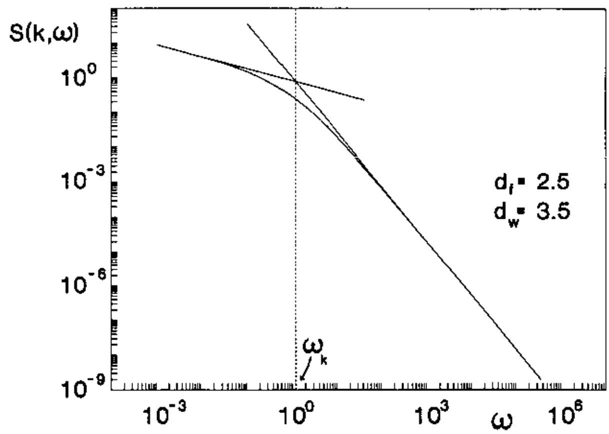

Figure 1. Scattering function $S(k,\omega)$ against ω for $k = 1$, $d_f = 2.5$ and $d_w = 3.5$. The straight lines have the asymptotic slopes predicted by (3.7) when $\omega \gg \omega_k$, and by (3.9) when $\omega \ll \omega_k$. The crossover frequency ω_k is indicated by the arrow.

We note that the low ω behaviour of $S(k,\omega)$ is still consistent with the sum rule (derived from the normalization condition of $P(r,t)$)

$$\lim_{k\to 0}\int_0^\infty d\omega\, S(k,\omega) = 1$$

since the weak divergengy (3.7) is integrable for $\omega \to 0$ ($\gamma < 1$), and for $\omega \to \infty$, $S(k,\omega)$ decays sufficiently fast ($\delta > 1$ in equation (3.6)).

Let us consider next the relaxation function, or intermediate scattering function, $P(k,t)$ which is defined as

$$P(k,t) = \int dr\, \exp(ik\cdot r)P(r,t) \tag{3.9}$$

whose Laplace transform is $S(k,s)$ (equation (3.2)). The scaling behaviour of $P(k,t)$ can be obtained from (3.4),

$$P(k,t) = f(k^{d_w}t) \tag{3.10}$$

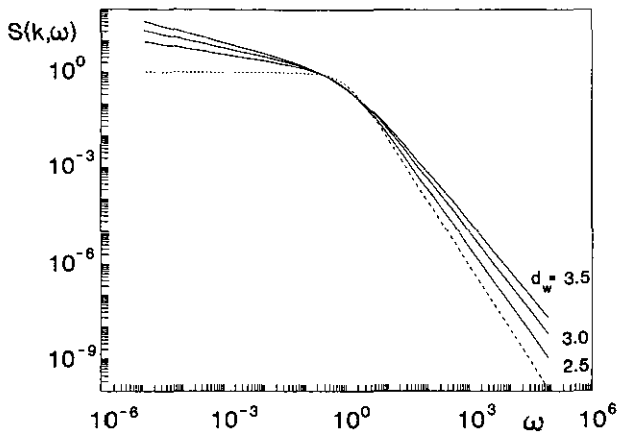

Figure 2. Scattering function $S(k,\omega)$ against ω for $k = 1$, $d_f = 2.5$ and several values of d_w. From top to bottom: $d_w = 3.5$, 3.0, and 2.5. The broken line corresponds to the Lorentzian result, $d_f = 3$ and $d_w = 2$.

where $f(x)$ is the scaling function. Similarly as for the one-dimensional case (see figure 1 in [1]), the functional form of $f(x)$ changes from an exponential behaviour at small $x \ll x_c$, to a power-law dependence when $x \gg x_c$. In one dimension, the 'crossover' value $x_c \to \infty$ when $d_w \to 2$ (d_f already sticks at its Euclidean value, $d_f = 1$) and, in this limit, $f(x) = \exp(-x)$ for all x. In the three-dimensional case, x_c diverges when both d_w and d_f tend to their Euclidean values. The asymptotic behaviour of $f(x)$ when $x \gg x_c$, can be obtained from (3.4) when $\beta \to 0$. We find

$$f(x) \sim \frac{1}{x^{1-\gamma}} \qquad x \gg x_c \tag{3.11}$$

where $\gamma = 1 - (d_f - \kappa)/d_w$ (equation (3.7*b*)). This power-law relaxation at large times is typical of equation (2.2). Even if $P(r,t)$ is 'well-behaved' at the origin, which is for instance the case when $\kappa = 0$ (see equation (2.9)), the slowing down (3.11) is still manifested.

The small x behaviour of $f(x)$ (or short-time behaviour of $P(k,t)$) can be obtained from (3.4) when $\beta \to \infty$,

$$S(k,s) \sim \frac{B}{s} - \frac{Ck^2}{s^{1+2/d_w}} + \cdots$$

where $B, C > 0$, yielding

$$f(x) \sim 1 - \text{constant } x^{2/d_w} \qquad x \ll x_c \tag{3.12}$$

a result that can be obtained easily from (3.9) when $k \to 0$ and $\mathrm{Re}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \cong 1-(\boldsymbol{k}\cdot\boldsymbol{r})^2/2$ holds. Note that (3.12) can be described approximately by the stretched exponential behaviour in time

$$P(k,t) \sim \exp[-\text{constant } (k^{d_w} t)^{2/d_w}] \qquad x \ll x_c$$

which corresponds to the Euclidean limit, $P(k,t) = \exp(-D_0 k^2 t)$, when $x_c \to \infty$ and $d_w \to 2$.

4. Summary and concluding remarks

We have discussed in detail a general form of a fractional differential equation which suitably describes the asymptotic behaviour of the probability density $P(r,t)$ for diffusion in homogeneous fractal structures. The generalized fractional diffusion equation (FDE) constitutes a natural extension of the FDE valid in Euclidean systems. The solution of the general FDE is obtained in a close integral form and the asymptotic behaviours of $P(r,t)$ for both small and large distances r are obtained exactly.

For distances r large compared with the diffusion length $R \sim t^{1/d_w}$, we obtain the stretched Gaussian form

$$P(r,t) \sim P_0(t)\exp[-a(r/R)^u] \qquad r/R \gg 1$$

with $u = d_w/(d_w - 1)$, while for short distances

$$P(r,t) \sim P_0(t)(r/R)^{-\kappa} \qquad r/R \ll 1$$

with $\kappa = (d_s - 1)/2 < 1$, where the spectral dimension $d_s = 2d_f/d_w$ determines the behaviour of the normalization factor, $P_0(t) \sim t^{-d_s/2}$, and d_f is the fractal dimension.

Accordingly, the relaxation function $P(k,t)$ associated with $P(r,t)$ displays a power-law behaviour at long times

$$P(k,t) \sim \frac{1}{(k^{d_w}t)^{1-\gamma}} \qquad k^{d_w}t \gg 1$$

where $\gamma = 1 - (d_f - \kappa)/d_w$. At short times,

$$P(k,t) \sim \exp(-\text{constant } k^2 t^{2/d_w}) \qquad k^{d_w}t \ll 1$$

displaying the known stretched exponential time dependence.

An exact expression is obtained for the dynamic structure factor $S(k,\omega)$. For large frequencies, we find

$$S(k,\omega) \sim \frac{k^2}{\omega^{1+2/d_w}} \qquad \omega \gg k^{d_w}$$

while at low frequencies,

$$S(k,\omega) \sim \frac{1}{\omega^{\gamma}}\frac{1}{k^{d_w(1-\gamma)}} \qquad \omega \ll k^{d_w}.$$

Some questions, however, remain open and further theoretical and numerical work is required to clarify them. In particular, the behaviour of $P(r,t)$ near the origin, $r/R \ll 1$, represents one of the most intriguing aspects deriving from the present approach. Understanding this problem is also important for determining the behaviour of the relaxation function $P(k,t)$ at long times, $k^{d_w}t \gg 1$. Further extensions of this approach can be achieved by studying the whole family of FDE as briefly discussed in [1]. Perhaps, such an attempt may lead us to a complete and satisfactory theory of diffusion on fractal structures.

Acknowledgment

We wish to thank A R Giona for fruitful discussions.

Appendix A

To obtain the asymptotic behaviour of the probability density in the temporal domain, we assume the scaling form (see also [1])

$$P(r,t) = \frac{a}{t^{d_s/2}}\left(\frac{r}{R}\right)^{\alpha} \exp[-b(r/R)^u] \qquad r/R \gg 1 \tag{A1}$$

where $a, b > 0$ are constants, and determine the exponents α and u such that the Laplace transform of (A1) is consistent with (2.3). The Laplace transform of $P(r,t)$

$$P(r,s) = \int_0^\infty \mathrm{d}t \, \exp(-st) P(r,t) \tag{A2}$$

can be evaluated by steepest descent methods. The leading contribution to (A2) when $r \to \infty$ and $s \to 0$, occurs for times t close to the value t_* which minimizes the argument $\rho(t)$ of the exponential factors in the integrand, $\rho(t) = st + b(r/R)^u$. Thus, we find

$$t_* \sim (rs^{-1/d_w})^{d_\phi} \qquad d_\phi = \frac{ud_w}{u + d_w}.$$

Expanding $\rho(t)$ up to second order terms $(t - t_*)^2$, (A2) becomes

$$P(r,s) \cong \frac{a}{t_*^{d_s/2}}\left(\frac{r}{R(t_*)}\right)^{\alpha} \exp[-\rho(t_*)] \int_0^\infty \mathrm{d}t \, \exp[-\rho''(t_*)(t - t_*)^2/2] \tag{A3}$$

where $\rho''(t_*)$ denotes the second derivative of ρ with respect to t evaluated at $t = t_*$, $\rho''(t_*) \sim s^2(rs^{1/d_w})^{u(d_w-1)/d_w-2}$. The remaining Gaussian integration in (A3) yields an extra factor $\sim [\rho''(t_*)]^{-1/2}[1 + \mathrm{erf}(t_*\sqrt{\rho''(t_*)/2})]$.

Comparison of the arguments of the exponentials in (A3) and (2.3) yields $d_\phi = 1$, which implies

$$u = \frac{d_w}{d_w - 1} \tag{A4}$$

and from the power law $(rs^{1/d_w})^{-\kappa}$ in (2.3), we find

$$\alpha = \frac{u}{2}(d_s - 1 - 2\kappa). \tag{A5}$$

Equation (A5) yields $\alpha = -d_f(d_w/2 - 1)/(d_w - 1) < 0$ when $\kappa = (d_f - 1)/2$, and $\alpha = 0$ when $\kappa = (d_s - 1)/2$. We now see one of the roles of the exponent κ: It determines the exponent α in the power-law prefactor in (A1) describing the asymptotic behaviour of $P(r,t)$ when $r/R \gg 1$ (see also appendix B).

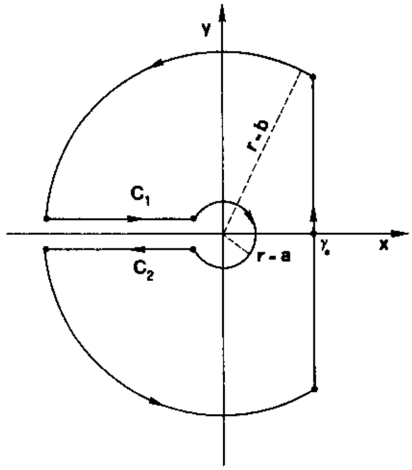

Figure 3. Contour of integration in the complex plane for evaluating the inverse Laplace transform formula (B1).

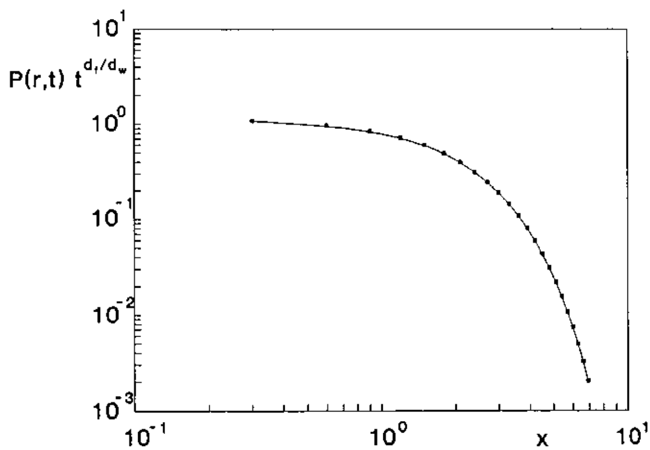

Figure 4. Probability density $P(r,t)t^{d_f/d_w}$ against $x \equiv r/R$, as obtained by numerical integration of (B2) when $d_f = 2.5$, $d_w = 3.5$, $\kappa = (d_s - 1)/2 = 0.214$, and $A' = \pi$. The full curve represents the analytical form $f(x) = (a + bx^{-\kappa})\exp(-c\ x^u)$, with $u = 1.4$, showing the two asymptotic behaviours $f(x) \sim x^{-\kappa}$ when $x \to 0$, and $f(x) \sim \exp(-cx^u)$ when $x \to \infty$. The constants a, b and c are fitting parameters.

Appendix B

The inverse Laplace transform of (2.3) can be evaluated from the complex inversion formula

$$P(r,t) = \frac{1}{2\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} e^{st} P(r,s)\, ds \qquad \text{(B1)}$$

following the integration path shown in figure 3. It can be shown that the integrals along the circular paths vanish and applying the theorem of Cauchy (B1) becomes

$$P(r,t) = - \lim_{a\to 0, b\to\infty} \frac{1}{2\pi \mathrm{i}} \left(\int_{C_1} \mathrm{d}s\, \mathrm{e}^{st} P(r,s) + \int_{C_2} \mathrm{d}s\, \mathrm{e}^{st} P(r,s) \right)$$

leading to the result

$$P(r,t) = \frac{A'}{\pi} \frac{1}{t^{d_f/d_w}} \left(\frac{r}{R}\right)^{-\kappa} \int_0^\infty \mathrm{d}x\, \frac{\exp[-x - (r/R)x^{1/d_w}\cos(\pi/d_w)]}{x^\gamma} \sin(\theta) \tag{B2}$$

where $R = t^{1/d_w}$, $\theta = \pi(1-\gamma) - (r/R)x^{1/d_w}\sin(\pi/d_w)$ and

$$\gamma = 1 - \frac{d_f - \kappa}{d_w}. \tag{B3}$$

Since $\kappa < d_f$ (see section 2), we see from (B3) that $\gamma < 1$ and the integral in (B2) converges for all r.

Numerical integration of (B2) in the case that $d_f = 2.5$, $d_w = 3.5$ and $\kappa = (d_s - 1)/2 \cong 0.214$ is shown in figure 4. For $r/R \gg 1$, (B2) coincides with the predicted stretched Gaussian behaviour (see appendix A) $P(r,t) \sim \exp[-b(r/R)^u]$, with $u = d_w/(d_w - 1) = 1.4$. For $r/R \ll 1$, (B2) develops the weak divergency $P(r,t) \sim (r/R)^{-\kappa}$.

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